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Global properties of the black cigar spacetime

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Abstract: In this work, we use parallel transport of vectors to study global characteristics of the Schwarzschild spacetime, in the context of the five-dimensional Randall-Sundrum braneworld, in the spacetime known as black cigar. We examine constant time circular orbits, where the parallel transport matrix indicates the existence of the analogue gravitational Aharonov-Bohm effect, except for some particular radii, where this matrix is trivial. Parallel transport along radial and time displacements are also obtained, reducing, in all cases to the usual results of ordinary Schwarzschild spacetime. On the other hand, the parallel transport matrix for a circular orbit in the Schwarzschild-AdS spacetime, an alternative solution for the black hole on the brane, keeps a "memory" of the higher dimension.

Keywords: D-branes, Global Symmetries, Classical Theories of Gravity, Black Holes.

Contents

1. Introduction

Topological defects play a fundamental role in many branches of physics. From vortices in high temperature superconductors to cosmic strings, they are everywhere. There is even the possibility that we live in such a defect, a domain wall in five dimensional anti-de Sitter space, or three-brane. This is the braneworld scenario of Randall and Sundrum (RS) [1, 2]. The idea of a higher dimensional universe is not new in physics going back to the old Kaluza-Klein theory, which merged electromagnetism and general relativity in a five dimensional model where the extra dimension was circular. Compact extra dimensions were the rule in the higher dimensional theories that appeared in the context of string theory. This makes the extra dimensions unobservable at low energies. As shown by RS, instead of compact extra dimensions these may be infinite but geometrically warped by means of a negative cosmological constant, such that most of the physical space is concentrated near the domain wall. In their model, the matter fields are confined to the brane leaving only gravity to propagate in the extra dimensions. The familiar four dimensional gravity appears as a consequence of the localization of the zero-mode of the Kaluza-Klein graviton on the brane. It is therefore very important to verify where four dimensional general relativity is affected by the extra dimensions.

In this work we study the global properties of the Schwarzschild black hole on the braneworld as obtained by Chamblin, Hawking and Reall (CHR) [3]. They have found evidence for a stable "cigar" shaped solution in five dimensions which, when intersecting the domain wall, induces the Schwarzschild metric. Since gravity propagates in all dimensions, a black hole on a brane will have its horizon extending out of the brane, being therefore a higher dimensional object. CHR showed that the Schwarzschild-AdS black hole in five dimensions can not intersect a vacuum domain wall because it fails to satisfy the Israel junction conditions, making it a poor candidate for the higher dimensional object looked for. A more likely candidate is a black string that extends the Schwarzschild singularity on the brane along the transverse direction. However this leads to curvature singularities at the AdS horizon. Also, the black string is unstable near the AdS horizon. CHR suggested that the black string horizon "pinches off" near the AdS horizon becoming a stable "black cigar". Near the brane, far from the AdS horizon, the metric is then approximated by the black string metric, which induces the Schwarzschild metric on the brane.

Our study of the global properties of the black cigar spacetime is done by computing the orthonormal frame matrix for the parallel transport of a vector along some specific paths. When this path is closed one obtains the holonomy matrix. When a vector is parallelly propagated along a loop in a manifold M, the curvature of the manifold causes the vector, initially at $p \in M$, to appear rotated with respect to its initial orientation in tangent space T_pM, when it returns to p. The holonomy is the path dependent linear transformation $T_pM \to T_pM$ responsible for this rotation. Positive and negative curvature manifolds, respectively, yield deficit or excess angles between initial and final vector orientation under parallel transport around such loops. This global property of the manifold can be used as a means of global classification of spacetimes, as pointed out by Rothman, Ellis and Murugan [4], who investigated the holonomy in the Drost-Schwarzschild geometry.

2. Black string in the RS scenario

RS first model (RS1) required two domain walls, one with negative tension and the other with positive tension, separated by AdS space, which being hyperbolic means that the domain walls are horospheres. In their second model (RS2), the negative tension wall is made to approach the AdS horizon and only one wall is considered. Notice that in RS1 the extra dimension is finite (the separation between the walls) whereas in RS2 it is infinite. In both cases the hierarchy problem is solved thanks to the geometry of AdS space.

In the models considered by Randall and Sundrum, the five-dimensional AdS metric is given by

$$ds^2 = e^{-2y/l}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \,, \tag{2.1}$$

where y is the extra dimension, $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric and l is the AdS radius. The AdS horizon is at $y = \infty$. In RS1 the positive and negative tension walls are, respectively, at $y = 0$ and $y = y_c$. In RS2 $y_c \to \infty$ and the negative tension wall is not considered.

With the transformation $z = le^{y/l}$ the AdS metric becomes

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{2.2}$$

The AdS horizon is at $z = \infty$ and timelike infinity at $z = 0$. In these coordinates the hyperbolic nature of AdS space is evident. If one looks at the two dimensional slice $ds^2 = \frac{l^2}{z^2}\left(dz^2 + dy^2\right)$ for instance, it is just Poincaré's upper half plane of two dimensional hyperbolic geometry. The points on the y axis ($z = 0$) are the points at infinity.

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Notice that the metric (2.2) is ostensively conformally flat. Replacing the Minkowski metric by any Ricci flat metric makes (2.2) still satisfy the Einstein equations with negative cosmological constant [5]. Using the four dimensional Schwarzschild metric as this Ricci flat metric CHR obtained the black string in five dimensional AdS:

$$ds^2 = \frac{l^2}{(|w| + z_0)^2} \left[-U(r)dt^2 + U(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dw^2 \right], \tag{2.3}$$

where the function $U(r)$ is given by

$$U(r) = 1 - \frac{2M}{r}. \tag{2.4}$$

The domain wall is at $z = z_0$ and $w = z - z_0$.

3. Parallel transport

The orthonormal frame matrix H for the parallel transport of a vector along a path γ, connecting points A and B in a manifold is given by the path-ordered exponential

$$H(\gamma) = \mathcal{P} \exp \left(- \int_A^B \Gamma_\mu dx^\mu \right), \tag{3.1}$$

where Γ_μ is the n-adic connection on the manifold. When the path γ is closed $H(\gamma)$ is known as the holonomy matrix. Notice the similarity of this equation with the Dirac phase factor $\Phi(\gamma) = \mathcal{P}\exp(ig \oint_\gamma A_\mu dx^\mu)$, the observable in the Aharonov-Bohm effect.

An important topological invariant, the Wilson loop [6], can be easily obtained from the holonomy $H(\gamma)$ on a closed path γ:

$$W(\gamma) = -n + Tr\,(H(\gamma)). \tag{3.2}$$

Holonomy has had an important role in the loop formulation of gauge theories [7, 6] and quantum gravity [8] and is a very convenient tool to obtain topological properties of specific geometries as used, for instance, by Bollini, Giambiagi and Tiomno [9] who investigated the Kerr black hole spacetime. In a recent article, Rothman, Ellis and Murugan [4] investigated the holonomy in the Drost-Schwarzschild geometry obtaining a quantized band structure of holonomy invariance when studying parallel transport around circular orbits. The clock effects were investigated by Bini, Jantzen and Mashhoon [10] via holonomy transformation. Burges [11] and Bezerra [12] examined the effects of parallel transport of vectors and spinors around a point-like solution and a cylindrically symmetric cosmic string, respectively, pointing out to a gravitational analogue of the Aharonov-Bohm effect. Two of us (C.F. and F.M.) [13] investigated, by holonomy transformation, the topological properties of a class of solutions in Kaluza-Klein theory and demonstrated that the holonomy gives a combined effect of gravitation and electromagnetic fields in the parallel transport of vectors. This combined the electromagnetic and gravitational Aharonov-Bohm effects. The analysis of Berry's quantum phase in gravitation [14] was also studied by this method in a recent article. The use of holonomy for quantum computation was analyzed in a geometric approach by Pachos and Zanardi [15].

4. Parallel transport in the RS2 black string scenario

We choose the following dual 1-form basis (co-frame) to describe the metric (2.3)

$$e^0 = \frac{l}{|w|+z_0} U(r)^{1/2} dt\,, \tag{4.1a}$$

$$e^1 = \frac{l}{|w|+z_0} U(r)^{-1/2} dr\,, \tag{4.1b}$$

$$e^2 = \frac{l}{|w|+z_0} r d\theta\,, \tag{4.1c}$$

$$e^3 = \frac{l}{|w|+z_0} r \sin\theta d\phi\,, \tag{4.1d}$$

$$e^4 = \frac{l}{|w|+z_0} dw. \tag{4.1e}$$

This way the metric (2.3) is given by $ds^2 = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2$.

The spin connections are found by solving the first Maurer-Cartan structure equation, $de^a + \omega^a_b \wedge e^b = 0$ for ω^a_b. From the basis (4.1) we find the following non null connections

$$\omega^2_4 = -\omega^4_2 = -r\zeta(w) d\theta\,, \tag{4.2a}$$

$$\omega^2_1 = -\omega^1_2 = \sqrt{U} d\theta\,, \tag{4.2b}$$

$$\omega^3_4 = -\omega^4_3 = -r\zeta(w)\sin\theta d\phi\,, \tag{4.2c}$$

$$\omega^3_1 = -\omega^1_3 = \sqrt{U}\sin\theta d\phi\,, \tag{4.2d}$$

$$\omega^3_2 = -\omega^2_3 = \cos\theta d\phi\,, \tag{4.2e}$$

$$\omega^1_4 = -\omega^4_1 = -\frac{\zeta(w)}{\sqrt{U}} dr\,, \tag{4.2f}$$

$$\omega^0_4 = \omega^4_0 = -\zeta(w)\sqrt{U} dt\,, \tag{4.2g}$$

$$\omega^0_1 = \omega^1_0 = \frac{U'}{2} dt\,, \tag{4.2h}$$

where the function $\zeta(w)$ is defined in terms of the Heaviside function

$$\zeta(w) = \frac{\Theta(w)}{|w|+z_0}\,. \tag{4.3}$$

The connection 1-forms result into the following spin connections

$$\Gamma_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{U}\sin\theta & 0 \\ 0 & 0 & 0 & -\cos\theta & 0 \\ 0 & \sqrt{U}\sin\theta & \cos\theta & 0 & -r\zeta\sin\theta \\ 0 & 0 & 0 & r\zeta\sin\theta & 0 \end{pmatrix}, \tag{4.4}$$

$$\Gamma_\theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{U} & 0 & 0 \\ 0 & \sqrt{U} & 0 & 0 & r\zeta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r\zeta & 0 & 0 \end{pmatrix}, \tag{4.5}$$

$$
\Gamma_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta/\sqrt{U} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta/\sqrt{U} & 0 & 0 & 0 \end{pmatrix}, \tag{4.6}
$$

and

$$
\Gamma_t = \begin{pmatrix} 0 & U'/2 & 0 & 0 & -\zeta\sqrt{U} \\ U'/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\zeta\sqrt{U} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}
$$

4.1 Constant time circular orbit

We begin by analyzing the case of a loop where the variables (t, r, θ) are constant, so the holonomy matrix is given simply in terms of the azimuthal spin connection

$$
H(\gamma_\phi) = \exp\left(-\oint \Gamma_\phi d\phi\right). \tag{4.8}
$$

Making the expansion of this equation and collecting the Γ_ϕ and Γ_ϕ^2 terms, we get

$$
H(\gamma_\phi) = 1 - \frac{\Gamma_\phi}{f_1(r,\theta,w)} \sin\left[2\pi f_1(r,\theta,w)\right] + \frac{(\Gamma_\phi)^2}{(f_1(r,\theta,w))^2}\left\{1 - \cos\left[2\pi f_1(r,\theta,w)\right]\right\}, \tag{4.9}
$$

with the function $f_1(r,\theta,w)$, defined by

$$
f_1(r,\theta,w) = \sqrt{[U + (r\zeta)^2]\sin^2\theta + \cos^2\theta}\,. \tag{4.10}
$$

Considering now only equatorial orbits, we set $\theta = \pi/2$. In this case the holonomy matrix is given by

$$
H(\gamma_\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{1-\cos 2\pi f_1}{f_1^2} U & 0 & \frac{\sin 2\pi f_1}{f_1^2}\sqrt{U} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{\sin 2\pi f_1}{f_1}\sqrt{U} & 0 & 1 - \frac{1-\cos 2\pi f_1}{f_1^2} U & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}
$$

Analyzing orbits in the brane; that is, making $w = 0$, we obtain $\zeta(0) = 0$, so the holonomy matrix is reduced to

$$
H(\gamma_\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos 2\pi\sqrt{U} & 0 & \sin 2\pi\sqrt{U} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin 2\pi\sqrt{U} & 0 & \cos 2\pi\sqrt{U} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.12}
$$

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As expected, the matrix obtained in this limit coincides with the correspondent holonomy matrix for ordinary four-dimensional Schwarzschild geometry. This is an important indication that the extra dimension probably does not leave a substantial imprint on the brane, at least that could be detected with parallel transport.

Notice that when $r \to \infty$ the holonomy matrix becomes trivial, yielding no deficit angle in the parallel transport of a vector around the loop. That is not the only situation where this occurs. For finite r, when

$$2\pi n f_1\left(r, \frac{\pi}{2}, w\right) = 2m\pi\,, \tag{4.13}$$

whith n, m integers, the holonomy is again the identity matrix. This implies that, for some critical radii r_{cr}, after n loops, the transported vector does not acquire a deficit angle. The critical radii are found by solving the equation below

$$\zeta^2 r_{cr}^3 + r_{cr}\xi(m,n) - 2M = 0\,, \tag{4.14}$$

with $\xi(m,n) = 1 - (m/n)^2$. In order that $r_{cr} > 0$ the function $\xi(m,n)$ must satisfy the inequality $\xi(m,n) > 0$ implying $m < n$. The real solution is given by

$$r_{cr} = -\frac{1}{3}\,\frac{-\left(27\,M\zeta + 3\sqrt{3}\sqrt{\xi^3 + 27\,M^2\zeta^2}\right)^{2/3} + 3\,\xi}{\zeta\,\sqrt[3]{27\,M\zeta + 3\sqrt{3}\sqrt{\xi^3 + 27\,M^2\zeta^2}}}\,. \tag{4.15}$$

Making $w = 0$, the critical radii on the brane are

$$r_{cr} = \frac{2M}{1 - (m/n)^2}. \tag{4.16}$$

This coincides with the result of Rothman, Ellis and Murugan [4] for ordinary four-dimensional Schwarzschild geometry.

The Wilson loop associated to the azimuthal holonomy matrix is given by

$$W(\gamma_\phi) = \mathrm{tr}\,H(\gamma_\phi) = -2\frac{1 - \cos 2\pi\sqrt{U + (r\zeta)^2}}{U + (r\zeta)^2}\,U\,. \tag{4.17}$$

In the limit $w \to 0$, the Wilson loop becomes

$$W(\gamma_\phi) = \mathrm{tr}\,H(\gamma_\phi) = -2 + 2\cos 2\pi\sqrt{U}\,, \tag{4.18}$$

which is the usual value for the Schwarzschild geometry. Notice that for $r \to \infty$ we have $W(\gamma) = 0$ in accord with the asymptotic flatness of the Schwarzschild spacetime.

4.2 Radial displacement

Another situation of interest corresponds to keeping the parameters (t, θ, ϕ) fixed, thus we have only the contribution of the radial connection to the H matrix

$$H(\gamma_r) = \exp\left(-\int_{r_1}^{r_2} \Gamma_r dr\right), \tag{4.19}$$

where Γ_r depends on r, differently from the previous case where Γ_ϕ did not depend on ϕ. The parallel transport matrix, between r_1 and r_2 is therefore given by

$$H(\gamma_r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\zeta\psi & 0 & 0 & \sin\zeta\psi \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\sin\zeta\psi & 0 & 0 & \cos\zeta\psi \end{pmatrix}, \tag{4.20}$$

where $\psi = \int_{r_1}^{r_2}(1/\sqrt{U})dr$ and the limits of integration (r_1, r_2) considered are outside of the event horizon. On the brane, when $w = 0$, we notice that the radial parallel transport is trivial as it is in ordinary Schwarzschild geometry, which has $\Gamma_r = 0$ and, consequently, $H = 1$.

4.3 Time displacement

Now we consider a time displacement; that is, a path with (r, θ, ϕ) fixed. We have

$$H(\gamma_t) = \exp\left(-\int_0^T \Gamma_t dt\right). \tag{4.21}$$

After a few calculations, we get

$$H(\gamma_t) = 1 - \frac{\Gamma_t}{f_2(r,w)}\sinh\left[Tf_2(r,z)\right] + \frac{(\Gamma_t)^2}{(f_2(r,w))^2}\left\{\cosh\left[Tf_2(r,w)\right] - 1\right\}, \tag{4.22}$$

with

$$f_2(r,w) = \sqrt{\left(\frac{U'^2}{4} + \zeta^2 U\right)}. \tag{4.23}$$

Finally, the matrix associated to the time displacement in five dimensional spacetime, is given by

$$H(\gamma_t) = \begin{pmatrix} \cosh Tf_2 & -\frac{\sinh Tf_2}{f_2}a & 0 & 0 & \frac{\sinh Tf_2}{f_2}b \\ -\frac{\sinh Tf_2}{f_2}a & 1 + \frac{\cosh Tf_2 - 1}{f_2^2}a^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{\sinh Tf_2}{f_2}b & -\frac{\cosh Tf_2 - 1}{f_2^2}ab & 0 & 0 & 1 + \frac{\cosh Tf_2 - 1}{f_2^2}b^2 \end{pmatrix}, \tag{4.24}$$

where $a = U'/2$ and $b = \zeta\sqrt{U}$.

When we restrict our analysis to paths on the brane, the coefficient b vanishes, so the time displacement matrix assumes the form of a Lorentz boost in Minkowski spacetime

$$H(\gamma_t) = \begin{pmatrix} \cosh TU'/2 & -\sinh TU'/2 & 0 & 0 & 0 \\ -\sinh TU'/2 & \cosh TU'/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.25}$$

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5. Holonomy for a circular orbit in Schwarzschild-AdS spacetime

An alternative candidate for a black hole on the brane is the Schwarzschild-AdS solution, which extends its horizon into the transverse dimensions to the brane. In this section we compute the holonomy for a constant time equatorial circular orbit in this spacetime, for comparison with the equivalent orbit in the black cigar spacetime. The Schwarzschild-AdS metric is given by [3]

$$ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + r^2\left(d\chi^2 + \sin^2\theta d\Omega^2\right), \tag{5.1}$$

where

$$U = 1 - \frac{2M}{r} + \frac{l^2}{r^2}. \tag{5.2}$$

The metric (5.1) may be described by the following dual 1-form basis

$$e^0 = U(r)^{1/2}dt\,, \tag{5.3a}$$
$$e^1 = U(r)^{-1/2}dr\,, \tag{5.3b}$$
$$e^2 = rd\chi\,, \tag{5.3c}$$
$$e^3 = r\sin\chi d\theta\,, \tag{5.3d}$$
$$e^4 = r\sin\chi\sin\theta d\phi\,, \tag{5.3e}$$

resulting into the azimuthal spin connection

$$\Gamma_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{U}\sin\chi\sin\theta \\ 0 & 0 & 0 & 0 & -\cos\chi\sin\theta \\ 0 & 0 & 0 & 0 & -\cos\theta \\ 0 & \sqrt{U}\sin\chi\sin\theta & \cos\chi\sin\theta & \cos\theta & 0 \end{pmatrix}. \tag{5.4}$$

We are interested in constant time (t fixed) equatorial ($\theta = \pi/2$) circular (r fixed) orbits on the brane ($\chi = \pi/2$). In this case

$$\Gamma_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{U} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{U} & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

The holonomy is therefore given by the matrix

$$H_{\gamma_\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos 2\pi\sqrt{U} & 0 & 0 & \sin 2\pi\sqrt{U} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\sin 2\pi\sqrt{U} & 0 & 0 & \cos 2\pi\sqrt{U} \end{pmatrix}, \tag{5.6}$$

which agrees with the black cigar case (4.12) in its form but, notice that U is different. Here it keeps the "memory" of the AdS background in the $\frac{l^2}{r^2}$ term in expression (5.2).

6. Concluding remarks

The parallel transport matrix $H(\gamma)$ gives the parallel transport of a vector along a given path γ. It is frequently used as a means of studying global characteristics of spacetimes. In this article we use it to investigate the black cigar five dimensional spacetime, whose four dimensional section $w = 0$ is Schwarzschild spacetime. When analysing constant time circular orbits we find some critical radii, where the matrix is trivial, implying that the parallel transport of a vector along such orbit does not change its orientation. Comparing with the Aharonov-Bohm (AB) phase $\oint A_\mu dx^\mu$, the gravitational analogue AB effect [12] occurs when the holonomy $\oint \Gamma_\mu dx^\mu$ is nontrivial. Therefore, there is a analogue AB effect in the black cigar spacetime, except at those critical radii. Off the brane, when $w \neq 0$, in the limit $M \to 0$, the holonomy is not trivial, except for those critical radii. On the brane, at $w = 0$, the holonomy is trivial for this limit as it is in ordinary four-dimensional Schwarzschild spacetime. The Wilson loop was calculated for these orbits, again reducing to its usual Schwarzschild value.

We also investigated the parallel transport matrix along a radial displacement, where it resembles a rotation matrix. On the brane, this matrix becomes trivial. Again, as it is in the usual Schwarzschild spacetime. For a time displacement the parallel transport matrix reduces to a boost-like matrix on the brane.

The holonomy for a constant time equatorial circular orbit in Schwarzschild-AdS spacetime was also computed for comparison with the equivalent black cigar spacetime orbit. The result is quite similar except that in the first case a "memory" of the extra dimension appears in the result making gravity on the brane not the same as the usual four-dimensional Schwarzschild solution. In any case, as shown by CHR, the Schwarzschild-AdS can not be intersected by a vacuum domain wall, making it a poor candidate for the black hole on the brane.

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References

[1] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [hep-ph/9905221].

[2] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [hep-th/9906064].

[3] A. Chamblin, S.W. Hawking and H.S. Reall, *Brane-world black holes*, *Phys. Rev.* **D 61** (2000) 065007 [hep-th/9909205].

[4] T. Rothman, G.F.R. Ellis and J. Murugan, *Holonomy in the Schwarzschild-Droste geometry*, *Class. and Quant. Grav.* **18** (2001) 1217 [gr-qc/0008070].

[5] D. Brecher and M.J. Perry, *Ricci-flat branes*, *Nucl. Phys.* **B 566** (2000) 151 [hep-th/9908018].

[6] K.G. Wilson, *Confinement of quarks*, *Phys. Rev.* **D 10** (1974) 2445.

[7] S. Mandelstam, *Quantum electrodynamics without potentials*, *Ann. Phys. (NY)* **19** (1962) 1; *Feynman rules for electromagnetic and Yang-Mills fields from the gauge independent field theoretic formalism*, *Phys. Rev.* **175** (1968) 1580.

[8] A. Ashtekar, *Non-perturbative canonical gravity*, World Scientific, Singapore, 1990.

[9] C.G. Bollini, J.J. Giambiagi and J. Tiomno, *Wilson loops in kerr gravitation*, *Nuovo Cim. Lett.* **31** (1981) 13.

[10] D. Bini, R.T. Jantzen and B. Mashhoon, *Circular holonomy and clock effects in stationary axisymmetric spacetimes*, *Class. and Quant. Grav.* **19** (2002) 17 [gr-qc/0111028].

[11] C.J.C. Burges, *The gravitational Aharonov-Bohm effect in three- dimensions*, *Phys. Rev.* **D 32** (1985) 504.

[12] V.B. Bezerra, *Gravitational analog of the Aharonov-Bohm effect in four-dimensions and three-dimensions*, *Phys. Rev.* **D 35** (1987) 2031.

[13] J.G. de Assis, C. Furtado, V.B. Bezerra and F. Moraes, *Loop variables for a class of space-times in Kaluza-Klein theory*, *Grav. Cosmol.* **6** (2000) 233.

[14] J.G. de Assis, C. Furtado and V.B. Bezerra, *Gravitational Berry's quantum phase*, *Phys. Rev.* **D 62** (2000) 045003.

[15] J. Pachos and P. Zanardi, *Quantum holonomies for quantum computing*, *Int. J. Mod. Phys.* **B 15** (2001) 1257 [quant-ph/0007110].